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Mellin-Barnes' type integrals and power series with the Riemann zeta-function in the coefficients*

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1 Introduction

The main aim of this article is to investigate the following two types of power series whose coefficients involve the Riemann zeta-function $\zeta(s)$. The first object is a binomial type series (2.1) given below, which will be studied in the next section, while the asymptotic behaviour of an exponential type series (3.2) will be investigated in Sections 3 and 4. Mellin-Barnes' type integral formula such as (2.2) and (3.3) will play central roles in both of these investigations. Furthermore, as for generalizations of these power series, we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties in the final section.

It should be noted that Mellin-Barnes' type integral formulae have been applied to deduce full asymptotic expansions for the mean squares of Dirichlet L-functions and Lerch zeta-functions (see [Ka1] and [Ka2]). The main method of the following derivation is based on a certain path shifting argument, which is similar to [Ka1][Ka2], for Mellin-Barnes' type integral formulae. Most of the results in this article, together with outline of proofs, have been announced in [Ka3].

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2 Binomial type series

Let $\alpha > 0$ be a parameter, and $\zeta(s,\alpha)$ the Hurwitz zeta-function defined by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}\qquad(\mathrm{Re}\, s>1),$$

and its meromorphic continuation over the whole s-plane. Let $\Gamma(s)$ be the gamma-function, and $(s)_n=\Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol.

A simple relation

$$\sum_{n=2}^{\infty}\{\zeta(n)-1\}=1,$$

which was firstly mentioned by Christian Goldbach in 1729 (see [Sr2, Section 1]), follows immediately from the inversion of the order of the double sum $\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}m^{-n}$. This is in fact derived as a special case of S. Ramanujan's formula

$$\zeta(\nu,1+x)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\zeta(\nu+n)(-x)^n \tag{2.1}$$

for $|x|<1$ and any complex $\nu\notin\{-1,0,1,2,\ldots\}$, which gives a base of his various evaluations of sums involving $\zeta(s)$ (see [Ram, Sections 5 and 6]). Noting the relations $\zeta(s,1)=\zeta(s)$ and $(\partial/\partial\alpha)^n\zeta(s,\alpha)=(-1)^n(s)_n\zeta(s+n,\alpha)$, we see that the right-hand side of (2.1) is actually the Taylor series expansion of $\zeta(\nu,1+x)$ as a function of x near $x=0$. H. M. Srivastava [Sr1][Sr2][Sr3] derived various summation formulae related to (2.1), while D. Klusch [Kl] considerd a generalization of (2.1) to the Lerch zeta-function. This direction has been further pursured by M. Yoshimoto, S. Kanemitsu and the author [YKK]. V. V. Rane [Ran] recently applied (2.1) to study the mean square of Dirichlet L-functions. For related results and various generalizations of (2.1), we refer to [Kl][Sr3] and their references.

For our later purpose we shall prove (2.1) as an application of Mellin-Barnes' type integrals. Suppose first that $\mathrm{Re}\,\nu>1$, and set

$$F_\nu(x)=\frac{1}{2\pi i}\int_{(b)}\frac{\Gamma(\nu+s)\Gamma(-s)}{\Gamma(\nu)}\zeta(\nu+s)x^s ds \tag{2.2}$$

for $x>0$, where b is a constant fixed with $1-\mathrm{Re}\,\nu<b<0$, and (b) denotes the vertical straight line from $b-i\infty$ to $b+i\infty$. We can shift the path of integration in (2.2) to the right, provided $0 < x < 1$, since the order of the integrand is $O\{x^{N+\frac{1}{2}}(|\operatorname{Im} s| + 1)^{\operatorname{Re}\nu-1}e^{-\pi|\operatorname{Im} s|}\}$ on the vertical line $\operatorname{Re} s = N + \frac{1}{2}$ $(N = 0, 1, 2, \ldots)$. Collecting the residues at the poles $s = n$ $(n = 0, 1, 2, \ldots)$, we see that $F_\nu(x)$ is equal to the right-hand infinite series in (2.1). On the other hand, since $\zeta(\nu + s) = \sum_{n=1}^{\infty} n^{-\nu-s}$ converges absolutely on the path $\operatorname{Re} s = b$, the term-by-term integration is permissible on the right-hand side of (2.2). Each term in the resulting expression can be evaluated by

$$(n+x)^{-\nu} = \frac{1}{2\pi i}\int_{(b)} \frac{\Gamma(-s)\Gamma(\nu+s)}{\Gamma(\nu)} n^{-\nu-s} x^s ds.$$

This can be obtained by taking $-z = x/n$ in

$$\Gamma(a)(1-z)^{-a} = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma(-s)\Gamma(a+s)(-z)^s ds$$

for $|\arg(-z)| < \pi$ and $-\operatorname{Re} a < \sigma < 0$, which is a special case of Mellin-Barnes' integral formula for Gauss' hypergeometric function (cf. [WW, p.289, 14.51, Corollary]). We therefore obtain

$$F_\nu(x) = \sum_{n=1}^{\infty}(n+x)^{-\nu} = \sum_{n=0}^{\infty}(n+1+x)^{-\nu} = \zeta(\nu, 1+x),$$

from which (2.1) immediately follows by analytic continuation.

3 Exponential type series

In 1962, S. Chowla and D. Hawkins [CH] found that the sum

$$G_0(x) = \sum_{n=2}^{\infty} \zeta(n)\frac{(-x)^n}{n!}$$

has the asymptotic formula

$$G_0(x) = x\log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-A\sqrt{x}}) \tag{3.1}$$

as $x \to +\infty$, where γ is Euler's constant and A is a certain positive constant. They conjectured that the error estimate in (3.1) cannot be essentially sharpened. Let a be an arbitrary fixed real number. R. G. Buschman and H. M. Srivastava [BS] introduced a more general formulation

$$G_a(x) = \sum_{n>a+1} \zeta(n-a)\frac{(-x)^n}{n!},$$

where n runs through all nonnegative integers with $n > a+1$, and studied its asymptotic behaviour as $x \to +\infty$. The special cases $a = -2, -1$ and 1 have been investigated by D. P. Verma [Ve], J. Tennenbaum [Te], and D. P. Verma and S. N. Prasad [VP], respectively.

Let ν be an arbitrary fixed complex number. It is in fact possible to treat a slightly general sum

$$G_\nu(x) = \sum_{n > \mathrm{Re}\,\nu+1} \zeta(n-\nu)\frac{(-x)^n}{n!}, \tag{3.2}$$

based on the formula

$$G_\nu(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma(-s)\zeta(s-\nu)x^s ds \tag{3.3}$$

for $x > 0$, where c is a constant fixed with $\mathrm{Re}\,\nu + 1 < c < [\mathrm{Re}\,\nu] + 2$. Here $[\mathrm{Re}\,\nu]$ denotes the greatest integer not exceeding $\mathrm{Re}\,\nu$. (3.3) can be proved by shifting the path (c) to the right, and collecting the residues at the poles $s = n$ $(n = [\mathrm{Re}\,\nu]+2, [\mathrm{Re}\,\nu]+3, \ldots)$ of the integrand, since the order of the integrand is $O\{x^{N+\frac{1}{2}}(N!)^{-1}e^{-\frac{\pi}{2}|\mathrm{Im}\,s|}\}$ on the vertical line $\mathrm{Re}\,s = N + \frac{1}{2}$ $(N = [\mathrm{Re}\,\nu]+1, [\mathrm{Re}\,\nu]+2, \ldots)$. While the main method of [BS] is Euler-Maclaurin's summation device, our treatment of (3.2) is due to a refinement of original [CH].

In the next section we shall first give a proof of

Theorem 1. *The following formulae hold for all $x \geq 1$.*

(i) *If $\nu \notin \{-1, 0, 1, 2, \ldots\}$,*

$$G_\nu(x) = \Gamma(-\nu-1)x^{\nu+1} - \sum_{n=0}^{[\mathrm{Re}\,\nu]+1} \zeta(n-\nu)\frac{(-x)^n}{n!} + \mathcal{G}_\nu(x); \tag{3.4}$$

(ii) *If $\nu \in \{-1, 0, 1, 2, \ldots\}$,*

$$G_\nu(x) = -\frac{(-x)^{\nu+1}}{(\nu+1)!}\left(\log x + 2\gamma - \sum_{n=1}^{\nu+1}\frac{1}{n}\right) - \sum_{n=0}^{\nu} \zeta(n-\nu)\frac{(-x)^n}{n!} + \mathcal{G}_\nu(x), \tag{3.5}$$

where the empty sum is to be considered as zero. Here $\mathcal{G}_\nu(x)$ is the error term satisfying the estimate

$$\mathcal{G}_\nu(x) = O(x^{-C}) \tag{3.6}$$

for any $C > 0$, where the implied constant depends only on C and ν.

Remark. This theorem refines the results in [BS].

S. Chowla and D. Hawkins suggested in [CH] that the error term in (3.1) is expressible in terms of 'almost' Bessel functions; however, it seems that the functions have not been precisely determined. Let $K_\nu(z)$ be the modified Bessel function of the third kind defined by

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\{I_{-\nu}(z) - I_\nu(z)\},$$

where

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m+\nu}$$

is the Bessel function with purely imaginary argument (cf. [Er2, p.5, 7.2.2(12) and (13)]). We can indeed show that $\mathcal{G}_\nu(x)$ has the Voronoï type summation formula (cf. [Iv, Chapter 3]) involving $K_{\nu+1}(z)$.

Theorem 2. *For any $x \geq 1$ we have*

$$\begin{aligned}\mathcal{G}_\nu(x) &= 2\left(\frac{x}{2\pi}\right)^{\frac{1}{2}(\nu+1)} \sum_{n=1}^{\infty} n^{-\frac{1}{2}(\nu+1)}\Big\{e^{-\frac{\pi i}{4}(\nu+1)}K_{\nu+1}(2e^{\frac{\pi i}{4}}\sqrt{2n\pi x}) \\ &\qquad + e^{\frac{\pi i}{4}(\nu+1)}K_{\nu+1}(2e^{-\frac{\pi i}{4}}\sqrt{2n\pi x})\Big\}.\end{aligned}$$

Let $(\nu, m) = \Gamma(\frac{1}{2}+\nu+m)/m!\Gamma(\frac{1}{2}+\nu-m)$ for any integer $m \geq 0$ be Hankel's symbol. Applying the asymptotic expansion

$$K_{\nu+1}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left\{\sum_{m=0}^{M-1}(\nu+1, m)(2z)^{-m} + O(|z|^{-M})\right\} \tag{3.7}$$

for $|\arg z| < 3\pi/2$, $|z| \geq 1$ and any integer $M \geq 0$ (cf. [Er2, p.24, 7.4.1(4)]), to Theorem 2, we can further prove

Corollary. *The asymptotic formula*

$$\begin{aligned}\mathcal{G}_\nu(x) &= \sqrt{2}\left(\frac{x}{2\pi}\right)^{\frac{1}{2}\nu+\frac{1}{4}} e^{-2\sqrt{\pi x}} \\ &\quad \times \left\{\sum_{m=0}^{M-1}(\nu+1, m)(32\pi x)^{-\frac{m}{2}}\cos\left(2\sqrt{\pi x} + \frac{\pi}{4}\left(\nu + \frac{3}{2} + m\right)\right) + O(x^{-\frac{M}{2}})\right\}\end{aligned}$$

holds for all $x \geq 1$ and all integers $M \geq 0$, where the implied constant depends only on ν.

Remark. This corollary gives an affirmative answer to the conjecture of S. Chowla and D. Hawkins mentioned above.

4 Proof of Theorems 1, 2 and Corollary

In this section we shall prove Theorems 1, 2 and Corollary.

Proof of Theorem 1. We may restrict our consideration to the case $\nu \notin \{-1, 0, 1, \ldots\}$, since other cases can be treated by taking limits in (3.4). Let C be a constant fixed arbitrary with $-C < \min(0, \operatorname{Re}\nu + 1)$. Then we can shift the path of integration in (3.3) from (c) to $(-C)$, since the order of the integrand is $O(|\operatorname{Im} s|^B e^{-\frac{\pi}{2}|\operatorname{Im} s|})$ as $\operatorname{Im} s \to \pm\infty$, where $B > 0$ is a constant depending only on $\operatorname{Re} s$ and $\operatorname{Re}\nu$. Collecting the residues at the poles $s = n$ $(n = 0, 1, \ldots, [\operatorname{Re}\nu] + 1)$ and $\nu + 1$, we obtain (3.4) with

$$\mathcal{G}_\nu(x) = \frac{1}{2\pi i}\int_{(-C)} \Gamma(-s)\zeta(s-\nu)x^s ds. \tag{4.1}$$

The estimate (3.6) immediately follows by noting that $|x^s| = x^{-C}$ holds on the path $\operatorname{Re} s = -C$. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Here we fix C such as $-C < \min(0, \operatorname{Re}\nu)$. Substituting the functional equation $\zeta(s-\nu) = \chi(s-\nu)\zeta(1-s+\nu)$ (cf. [Iv, Chapter 1, p.9, 1.2(1.24)]) into the right-hand side of (4.1), we get

$$\begin{aligned}\mathcal{G}_\nu(x) &= \frac{x^{\nu+1}}{2\pi i}\int_{(-C)} \Gamma(-s)\Gamma(1-s+\nu)2\cos\left(\frac{\pi}{2}(s-\nu-1)\right)\\ &\quad \times \zeta(1-s+\nu)(2\pi x)^{s-\nu-1}ds.\end{aligned} \tag{4.2}$$

Since $\zeta(1-s+\nu) = \sum_{n=1}^\infty n^{s-\nu-1}$ converges absolutely on the path $\operatorname{Re} s = -C$, the term-by-term integration is permissible on the right-hand side of (4.2), and this gives

$$\mathcal{G}_\nu(x) = x^{\nu+1}\sum_{n=1}^\infty \left\{g_\nu(2n\pi x e^{\frac{\pi i}{2}}) + g_\nu(2n\pi x e^{-\frac{\pi i}{2}})\right\},$$

where

$$g_\nu(z) = \frac{1}{2\pi i}\int_{(-C)} \Gamma(-s)\Gamma(1-s+\nu)z^{s-\nu-1}ds \tag{4.3}$$

for $|\arg z| < \pi$. Noting that the pair

$$x^\nu K_\nu(x), \qquad 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)$$

for $\mathrm{Re}\, s > \max(0, -2\nu)$ is Mellin transforms (see [Ti, Chapter VII, p.197, (7.9.11)]), we immediately obtain

$$g_\nu(z) = 2z^{-\frac{1}{2}(\nu+1)} K_{\nu+1}(2z^{\frac{1}{2}})$$

for $|\arg z| < \pi$, by which the proof of Theorem 2 is complete. □

Proof of Corollary. From (3.7) with $M = 0$, we have

$$K_{\nu+1}(2e^{\pm\frac{\pi i}{4}}\sqrt{2n\pi x}) = O\{(nx)^{-\frac{1}{4}} \exp(-2\sqrt{n\pi x})\} \tag{4.4}$$

for $n = 1, 2, \ldots$ and $x \geq 1$. Noting that the inequality $\sqrt{n} \geq \sqrt{2}\left(1 + 5^{-1}\sqrt{n-2}\right)$ holds for all $n \geq 2$, we obtain

$$\sum_{n\geq 2} n^{-\frac{1}{2}(\nu+1)}(nx)^{-\frac{1}{4}} \exp(-2\sqrt{n\pi x}) = O\{x^{-\frac{1}{4}} \exp(-2\sqrt{2\pi x})\}.$$

This, together with Theorem 2 and (4.4), yield

$$\begin{aligned} \mathcal{G}_\nu(x) &= 2\left(\frac{x}{2\pi}\right)^{\frac{1}{2}(\nu+1)} \left\{e^{-\frac{\pi i}{4}(\nu+1)} K_{\nu+1}(2e^{\frac{\pi i}{4}}\sqrt{2\pi x}) + e^{\frac{\pi i}{4}(\nu+1)} K_{\nu+1}(2e^{-\frac{\pi i}{4}}\sqrt{2\pi x})\right\} \\ &\quad + O\{x^{\frac{1}{2}\mathrm{Re}\,\nu + \frac{1}{4}} \exp(-2\sqrt{2\pi x})\}, \end{aligned} \tag{4.5}$$

where the implied constant depends only on ν. The corollary now follows by substituting (3.7) into the first term on the right-hand side of (4.5). □

5 Generating functions of $\zeta(s)$

Let α and ν be arbitrary complex numbers with $\nu \notin \{1, 0, -1, \ldots\}$. We define

$$\begin{aligned} f_\nu(\alpha; z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}\zeta(\nu+n)z^n \quad (|z| < 1), \\ e_\nu(z) &= \sum_{n=0}^{\infty} \frac{1}{n!}\zeta(\nu+n)z^n \quad (|z| < +\infty). \end{aligned}$$

Since $\zeta(\nu+n) \to 1$ uniformly for $n = 0, 1, 2, \ldots$, as $\mathrm{Re}\,\nu \to +\infty$, we see that $f_\nu(\alpha; z) \to (1-z)^{-\alpha}$ and $e_\nu(z) \to e^z$, as $\mathrm{Re}\,\nu \to +\infty$. This suggests us to define the hypergeometric type generating functions of $\zeta(s)$ as

$$\mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\zeta(\nu+n)z^n \quad (|z| < 1), \tag{5.1}$$

$$\mathcal{F}_\nu(\alpha; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!}\zeta(\nu+n)z^n \quad (|z| < +\infty), \tag{5.2}$$

where α, β and γ are arbitrary fixed complex numbers with $\gamma \notin \{0, -1, -2, \ldots\}$. Then we can observe, when $\operatorname{Re}\nu \to +\infty$, that

$$\begin{aligned} \mathcal{F}_\nu(\alpha, \beta; \gamma; z) &\longrightarrow F(\alpha, \beta; \gamma; z), \\ \mathcal{F}_\nu(\alpha; \gamma; z) &\longrightarrow F(\alpha; \gamma; z), \end{aligned}$$

where $F(\alpha, \beta; \gamma; z)$ and $F(\alpha; \gamma; z)$ denote hypergeometric functions of Gauss and Kummer, respectively.

Substituting the series representation $\zeta(\nu + n) = \sum_{m=1}^{\infty} m^{-\nu-n}$ for $\operatorname{Re}\nu > 1$ and $n \geq 0$ into (5.1) and (5.2), and changing the order of summations, respectively, we get

Theorem 3. *The Dirichlet series expressions*

$$\mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \sum_{m=1}^{\infty} F\left(\alpha, \beta; \gamma; \frac{z}{m}\right) m^{-\nu}, \tag{5.3}$$

and

$$\mathcal{F}_\nu(\alpha; \gamma; z) = \sum_{m=1}^{\infty} F\left(\alpha; \gamma; \frac{z}{m}\right) m^{-\nu} \tag{5.4}$$

hold for $\operatorname{Re}\nu > 1$, *respectively.*

Recall that the hypergeometric functions have Euler's integral formulae (cf. [Er1, p.59, 2.1.3, (10), and p.255, 6.5, (1)]). Corresponding to these, from the term-by-term integrations, we can deduce

Theorem 4. *It follows that*

$$\mathcal{F}_\nu(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 \tau^{\beta-1}(1-\tau)^{\gamma-\beta-1} f_\nu(\alpha; \tau z)\, d\tau \tag{5.5}$$

for $0 < \operatorname{Re}\beta < \operatorname{Re}\gamma$ *and* $|z| < 1$, *and*

$$\mathcal{F}_\nu(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\gamma-\alpha-1} e_\nu(\tau z)\, d\tau \tag{5.6}$$

for $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma$ *and* $|z| < +\infty$.

Recall further that the hypergeometric functions have Mellin-Barnes' integral formula (cf. [Er1, p.62, 2.1.3, (15), and p.256, 6.5, (4)]). By the same path shifting argument as in Section 2, we can show

Theorem 5. *For* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ *and* $\operatorname{Re}\nu > 1$ *we have*

$$\mathcal{F}_\nu(\alpha,\beta;\gamma;z) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_{(b)}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}\zeta(\nu+s)(-z)^s ds, \tag{5.7}$$

for $|\arg(-z)| < \pi$, *where* b *is fixed with* $\max(-\operatorname{Re}\alpha, -\operatorname{Re}\beta, 1-\operatorname{Re}\nu) < b < 0$, *and*

$$\mathcal{F}_\nu(\alpha;\gamma;z) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\int_{(c)}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)}\zeta(\nu+s)(-z)^s ds \tag{5.8}$$

for $|\arg(-z)| < \pi/2$, *where* c *is fixed with* $\max(-\operatorname{Re}\alpha, 1-\operatorname{Re}\nu) < c < 0$.

Formulae (5.1)–(5.8) are fundamental in deriving various properties of $\mathcal{F}_\nu(\alpha,\beta;\gamma;z)$ and $\mathcal{F}_\nu(\alpha;\gamma;z)$. Further investigations and detailed proofs will be given in forthcoming papers.

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